Quantum Mechanics Derived from Boltzmann's Equation for the Planck Aether

F. Winterberg

Desert Research Institute, University and Community College System of Nevada, Reno, Nevada, USA

Z. Naturforsch. **50a**, 601–605 (1995); received December 13, 1994

In the Planck aether hypothesis it is assumed that space is densely filled with an equal number of locally interacting positive and negative Planck masses obeying a nonrelativistic law of motion. If described by a quantum mechanical two-component nonrelativistic nonlinear operator field equation, this model has a spectrum of particles greatly resembling the particles of the standard model, with Lorentz invariance as a derived dynamical symmetry valid in the limit of energies small compared to the Planck energy.

Here we show that quantum mechanics itself can be derived from the Newtonian mechanics of the Planck aether as an approximate solution of the Boltzmann equation for the positive and negative Planck masses, with departures from quantum mechanics suppressed by the Planck length.

In the *Planck aether hypothesis*, it is assumed that space is densely filled with an equal number of locally interacting positive and negative Planck masses obeying a nonrelativistic law of motion. In the *Planck aether model* it is assumed that this assembly can be described by the nonrelativistic Heisenberg-type operator field equation [1]

$$i\hbar \frac{\partial \psi_\pm}{\partial t} = \mp \frac{\hbar^2}{2m_p} \nabla^2 \psi_\pm + 2\hbar c r_p^2 (\psi_+^\dagger \psi_+ - \psi_-^\dagger \psi_-)\psi_\pm , \quad (1)$$

where $\psi_\pm$ are the field operators for the positive and negative Planck masses obeying the commutation relations

$$[\psi_\pm(\boldsymbol{r})\psi_\pm^\dagger(\boldsymbol{r}')] = \delta(\boldsymbol{r}-\boldsymbol{r}'),$$
$$[\psi_\pm(\boldsymbol{r})\psi_\pm(\boldsymbol{r}')] = [\psi_\pm^\dagger(\boldsymbol{r})\psi_\pm^\dagger(\boldsymbol{r}')] = 0, \quad (2)$$

and where r_p and m_p are the Planck length and mass derived from the relations $G m_p^2 = \hbar c$, $m_p r_p c = \hbar$, where G are Newton's and $\hbar$ Planck's constant. In the Planck aether each Planck mass occupies the volume r_p^3.

The purpose of the Planck aether model was to formulate a quantum field theory free of all divergencies and with a vanishing cosmological constant. The Planck aether model has a spectrum of particles greatly resembling the elementary particles of the standard model, with Lorentz invariance as a derived dynamic symmetry for energies small compared to the Planck energy. With the exception of the Planck masses, all particles are quantized collective excitations of the Planck aether. It is only these quasiparticles which obey Lorentz invariance as a dynamic symmetry. In contrast, the Planck masses are subject to Galilei invariance, seen here as the more fundamental kinematic symmetry. Because of this exceptional role, it would be more appealing if the Planck masses would be governed not just by a nonrelativistic Galilei invariant quantum mechanical law of motion, but by a Newtonian mechancial law of motion. The conjecture that quantum mechanics may be the result of Newtonian mechanics for the Planck aether, is supported by the fact that the fundamental force which can be constructed from G, c and $\hbar$, is given by $F = c^4/G$ and does not contain $\hbar$. Because the Planck aether model admits also negative masses, Newtonian mechanics for the Planck aether would have to be extended to negative masses. With the Planck masses only interacting locally, such a model, if successful, would describe the Planck aether solely by the kinetic energy of the Planck masses, with all forces reduced to kinematic boundary conditions at the surface of the colliding Planck masses. The model would be in line with Newton's idea that hard frictionless spheres are the ultimate building blocks of matter. Newton's system of hard frictionless spheres is probably the most perfect mechanical counterpart to Einstein's vacuum gravitational field equation as the most perfect description of a classical field. We now prove this conjecture.

Reprint requests to Prof. Dr. F. Winterberg.

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Taking the Hartree approximation of (1) by replacing the field operators with their expectation values, one obtains the one-body Schrödinger equation for a positive (or negative) Planck mass

$$i\hbar\frac{\partial\psi_\pm}{\partial t} = \mp\frac{\hbar^2}{2m_p}\nabla^2\psi_\pm + U(\boldsymbol{r})\psi_\pm \tag{3}$$

moving in the average potential

$$U(\boldsymbol{r}) = 2\hbar c r_p^2 \langle|\psi_+^\dagger\psi_+ - \psi_-^\dagger\psi_-|\rangle \tag{4}$$

produced by all the Planck masses. Through its kinetic energy term, the one-body Schrödinger equation alone implies that by going from classical mechanics to quantum mechanics, the particle momentum has to be replaced by the operator $p=(\hbar/i)/\partial q$. From there one obtains the commutation relation $[pq]=\hbar/i$, which for a field-theoretical treatment of the many-body problem leads to the commutation relations (2) for the field operators. It is for this reason sufficient to derive the one-body Schrödinger equation from the Planck aether hypothesis.

Within the Planck aether, a Planck mass is subject to collisions with Planck masses of equal and opposite sign. The collision between two negative Planck masses has the same outcome as the collision between two positive Planck masses, but this is not the case for the collision of a positive with a negative Planck mass. Even though the average effect of all Planck masses on one Planck mass can be described by a potential as in the Schrödinger equation (3), the collision of a positive with a negative Planck mass leads in addition to what Schrödinger [2] has called a "Zitterbewegung" (quivering motion). This can be seen as follows: As for the collision of Planck masses of equal sign, the tangential velocity components remain the same, but for the normal velocity components the outcome is different. If v'_+ and v'_- are the normal velocity components before, v_+ and v_- those after the collision, energy and momentum conservation imply that

$$v_+^2 - v_-^2 = v_+'^2 - v_-'^2,$$
$$v_+ - v_- = v'_+ - v'_-. \tag{5a, b}$$

Rewriting (5a) as $(v_+ - v_-)(v_+ + v_-) = (v'_+ - v'_-)(v'_+ + v'_-)$ and dividing it by (5b), one has

$$v_+ + v_- = v'_+ + v'_-. \tag{6}$$

From (5b) and (6) one has

$$v_+ = v'_+, \quad v_- = v'_-. \tag{7}$$

It thus follows that a collision between a positive and a negative Planck mass does not change the velocity neither in magnitude nor direction, but it permits a spatial parallel displacement of the trajectories. Expressed in terms of Planck's fundamental units, the displacement should be equal to

$$\delta = (1/2)\,a_p t_p^2, \tag{8}$$

where $a_p = F/m_p = c^4/G m_p = c^{7/2}/\sqrt{\hbar c}$ and $t_p = r_p/c$. One thus finds that

$$\delta = (1/2)\sqrt{\hbar G/c^3} = \pm(1/2)\,r_p = \pm\hbar/2m_p c, \tag{9}$$

which is just the "Zitterbewegung" radius derived by Schrödinger from the Dirac equation, with the "Zitterbewegung" velocity $a_p t_p = c$. We use this result to solve the Boltzmann equation for the Planck aether.

The Boltzmann equation is [3]

$$\frac{\partial f}{\partial t} + \boldsymbol{v}\cdot\frac{\partial f}{\partial \boldsymbol{r}} + \boldsymbol{a}\cdot\frac{\partial f}{\partial \boldsymbol{v}} = \int v_{\mathrm{rel}}(f'f'_1 - f f_1)\,d\sigma\, d\boldsymbol{v}_1, \tag{10}$$

where f is the distribution function of the colliding particles, f', f'_1 before and f, f_1 after the collision, with f'_1 and f_1 the distribution functions of the particles which by colliding with those particles belonging to f' and f change the distribution function from f' to f. The magnitude of the relative collision velocity is v_{rel}, and the collision cross section is σ. The particle number density is $n(\boldsymbol{r},t) = \int f(\boldsymbol{v},\boldsymbol{r},t)\,d\boldsymbol{v}$ and the average velocity $\boldsymbol{V} = \int \boldsymbol{v} f(\boldsymbol{v},\boldsymbol{r},t)\,d\boldsymbol{v}/n(\boldsymbol{r},t)$. The acceleration $\boldsymbol{a}$ is $\boldsymbol{a} = \mp(1/m_p)\nabla U$, where $U(\boldsymbol{r})$ is the potential of a force.

Applied to the Planck aether, the Boltzmann equation for the distribution function $f_\pm$ of the positive and negative Planck masses is

$$\frac{\partial f_\pm}{\partial t} + \boldsymbol{v}_\pm\cdot\frac{\partial f_\pm}{\partial \boldsymbol{r}} \mp \frac{1}{m_p}\frac{\partial U}{\partial \boldsymbol{r}}\cdot\frac{\partial f_\pm}{\partial \boldsymbol{v}_\pm} = 4\alpha c r_p^2 \int (f'_\pm f'_\mp - f_\pm f_\mp)\,d\boldsymbol{v}_\mp, \tag{11}$$

where we have set $\sigma = (2r_p)^2 = 4r_p^2$ and $v_{\mathrm{rel}} = \alpha c$, with α a numerical factor. Because of (9) one has

$$f'_\pm(\boldsymbol{r}) = f_\pm(\boldsymbol{r} \pm \boldsymbol{r}_p/2), \tag{12}$$

where one has to average over all possible displacements and velocities of the "Zitterbewegung". Because the distribution function f' before the collision is set equal the displaced distribution function f, the direction of the "Zitterbewegung" motion is in the opposite direction of the displacement vector $\boldsymbol{r}_p/2$.

With (12) the integrand in the collision integral becomes

$$f'_\pm f'_\mp - f_\pm f_\mp = f_\pm\left(\boldsymbol{r} \pm \frac{\boldsymbol{r}_p}{2}\right) f_\mp\left(\boldsymbol{r} \mp \frac{\boldsymbol{r}_p}{2}\right) - f_\pm(\boldsymbol{r}) f_\mp(\boldsymbol{r}) . \quad (13)$$

Expanding $f_\pm\left(\boldsymbol{r} \pm \frac{\boldsymbol{r}_p}{2}\right)$ and $f_\mp\left(\boldsymbol{r} \mp \frac{\boldsymbol{r}_p}{2}\right)$ into a Taylor series

$$\begin{aligned} f_\pm\left(\boldsymbol{r} \pm \frac{\boldsymbol{r}_p}{2}\right) &= f_\pm \pm \frac{\boldsymbol{r}_p}{2} \cdot \frac{\partial f_\pm}{\partial \boldsymbol{r}} + \frac{\boldsymbol{r}_p^2}{8} \frac{\partial^2 f_\pm}{\partial \boldsymbol{r}^2} + \ldots , \\ f_\mp\left(\boldsymbol{r} \mp \frac{\boldsymbol{r}_p}{2}\right) &= f_\mp \mp \frac{\boldsymbol{r}_p}{2} \cdot \frac{\partial f_\mp}{\partial \boldsymbol{r}} + \frac{\boldsymbol{r}_p^2}{8} \frac{\partial^2 f_\mp}{\partial \boldsymbol{r}^2} + \ldots , \end{aligned} \quad (14)$$

one finds up to second order

$$\begin{aligned} f'_\pm f'_\mp - f_\pm f_\mp \simeq{}& \pm \frac{\boldsymbol{r}_p^2}{2} \cdot \left(f_\mp \frac{\partial f_\pm}{\partial \boldsymbol{r}} - f_\pm \frac{\partial f_\mp}{\partial \boldsymbol{r}}\right) \\ & - \frac{\boldsymbol{r}_p^2}{4} \frac{\partial f_\pm}{\partial \boldsymbol{r}} \cdot \frac{\partial f_\mp}{\partial \boldsymbol{r}} + \frac{\boldsymbol{r}_p^2}{8} \cdot \left(f_\mp \frac{\partial^2 f_\pm}{\partial \boldsymbol{r}^2} + f_\pm \frac{\partial^2 f_\mp}{\partial \boldsymbol{r}^2}\right) \end{aligned} \quad (15)$$

with higher order terms suppressed by the Planck length. How higher order terms can be taken into account is shown in the appendix. Because approximately $f_\mp(\boldsymbol{v}_\mp, \boldsymbol{r}, t) \simeq f_\pm(\boldsymbol{v}_\pm, \boldsymbol{r}, t)$, one has

$$\begin{aligned} f'_\pm f'_\mp - f_\pm f_\mp &\simeq \frac{\boldsymbol{r}_p^2}{4} \left(\frac{\partial f_\pm}{\partial \boldsymbol{r}}\right)^2 + \frac{\boldsymbol{r}_p^2}{4} f_\pm \frac{\partial^2 f_\pm}{\partial \boldsymbol{r}^2} \\ &= \left(\frac{\boldsymbol{r}_p}{2}\right)^2 f_\pm^2 \frac{\partial^2 \log f_\pm}{\partial \boldsymbol{r}^2} \simeq \left(\frac{\boldsymbol{r}_p}{2}\right)^2 f_\pm f_\mp \frac{\partial^2 \log f_\pm}{\partial \boldsymbol{r}^2} . \end{aligned} \quad (16)$$

To average the "Zitterbewegung" displacement over a sphere with a volume to surface ratio $(r_p/2)^3/(r_p/2)^2 = r_p/2$, (16) must be multiplied by the operator $(1/2)\boldsymbol{r}_p \cdot \partial/\partial \boldsymbol{r}$, and to average over the velocity of the "Zitterbewegung" it must in addition be multiplied by the operator $\boldsymbol{c} \cdot \partial/\partial \boldsymbol{v}_\pm$, with the vector $\boldsymbol{c}$ in opposite direction to $\boldsymbol{r}_p$.

Integrating (11) over $dv_\mp$ and setting $\int f_\mp \, d\boldsymbol{v}_\mp \simeq 1/2 r_p^3$, the number density of one Planck mass species in the undisturbed configuration of the Planck aether, one has

$$\begin{aligned} \frac{\partial f_\pm}{\partial t} + \boldsymbol{v}_\pm \cdot \frac{\partial f_\pm}{\partial \boldsymbol{r}} \mp \frac{1}{m_p} \frac{\partial U}{\partial \boldsymbol{r}} \cdot \frac{\partial f_\pm}{\partial \boldsymbol{v}_\pm} \\ = - \frac{\alpha c^2 \boldsymbol{r}_p^2}{4} \frac{\partial^2}{\partial \boldsymbol{v}_\pm \partial \boldsymbol{r}} \left(f_\pm \frac{\partial^2 \log f_\pm}{\partial \boldsymbol{r}^2}\right) . \end{aligned} \quad (17)$$

For an approximate solution of (17) one computes its zeroth and first moment. The zeroth moment is obtained by integrating (17) over $d\boldsymbol{v}_\pm$, with the result that

$$\frac{\partial n_\pm}{\partial t} + \frac{\partial (n_\pm \boldsymbol{V}_\pm)}{\partial \boldsymbol{r}} = 0 , \quad (18)$$

which is the continuity equation for the macroscopic quantities $n_\pm$ and $\boldsymbol{V}_\pm$. The first moment is obtained by multiplying (17) with $\boldsymbol{v}_\pm$ and integrating over $d\boldsymbol{v}_\pm$. Because the logarithmic dependence can be written with sufficient accuracy as $\partial^2 \log f_\pm / \partial \boldsymbol{r}^2 \simeq \partial^2 \log n_\pm / \partial \boldsymbol{r}^2$, one finds

$$\begin{aligned} \frac{\partial (n_\pm \boldsymbol{V}_\pm)}{\partial t} + \frac{\partial (n_\pm \boldsymbol{V}_\pm \cdot \boldsymbol{V}_\pm)}{\partial \boldsymbol{r}} = \mp \frac{n_\pm}{m_p} \frac{\partial U}{\partial \boldsymbol{r}} \\ + \frac{\alpha c^2 \boldsymbol{r}_p^2}{4} \frac{\partial}{\partial \boldsymbol{r}} \left(n_\pm \frac{\partial^2 \log n_\pm}{\partial \boldsymbol{r}^2}\right) . \end{aligned} \quad (19)$$

With the help of (18) this can be written as

$$\begin{aligned} \frac{\partial \boldsymbol{V}_\pm}{\partial t} + \boldsymbol{V}_\pm \cdot \frac{\partial \boldsymbol{V}_\pm}{\partial \boldsymbol{r}} = \mp \frac{1}{m_p} \frac{\partial U}{\partial \boldsymbol{r}} \\ + \frac{\alpha \hbar^2}{4 m_p^2 n_\pm} \frac{\partial}{\partial \boldsymbol{r}} \left(n_\pm \frac{\partial^2 \log n_\pm}{\partial \boldsymbol{r}^2}\right) , \end{aligned} \quad (20)$$

for which one can also write

$$\begin{aligned} \frac{\partial \boldsymbol{V}_\pm}{\partial t} + \boldsymbol{V}_\pm \cdot \frac{\partial \boldsymbol{V}_\pm}{\partial \boldsymbol{r}} = \mp \frac{1}{m_p} \frac{\partial U}{\partial \boldsymbol{r}} \\ + \frac{\alpha \hbar^2}{2 m_p^2} \frac{\partial}{\partial \boldsymbol{r}} \left(\frac{1}{\sqrt{n_\pm}} \frac{\partial^2 \sqrt{n_\pm}}{\partial \boldsymbol{r}^2}\right) . \end{aligned} \quad (21)$$

The equivalence of (18) and (21) with the one-body Schrödinger equation (3) can now be established by Madelung's transformation

$$\begin{aligned} n_\pm &= \psi_\pm^* \psi_\pm , \\ n_\pm \boldsymbol{V}_\pm &= \mp \frac{i\hbar}{2 m_p} [\psi_\pm^* \nabla \psi_\pm - \psi_\pm \nabla \psi_\pm^*] \end{aligned} \quad (22)$$

transforming Schrödinger's equation (3) into

$$\begin{aligned} \frac{\partial n_\pm}{\partial t} + \frac{\partial (n_\pm \boldsymbol{V}_\pm)}{\partial \boldsymbol{r}} &= 0 , \\ \frac{\partial \boldsymbol{V}_\pm}{\partial t} + \boldsymbol{V}_\pm \cdot \frac{\partial \boldsymbol{V}_\pm}{\partial \boldsymbol{r}} &= \mp \frac{1}{m_p} \frac{\partial}{\partial \boldsymbol{r}} [U + Q_\pm] , \end{aligned} \quad (23)$$

where

$$Q_\pm = \mp \frac{\hbar^2}{2 m_p} \frac{1}{\sqrt{n_\pm}} \frac{\partial^2 \sqrt{n_\pm}}{\partial \boldsymbol{r}^2} \quad (24)$$

is the so called quantum potential. By comparison with (18) and (21) one finds full equivalence for $\alpha = 1$, that is for $v_{\text{rel}} = c$.

We have shown that the Schrödinger equation for a Planck mass m_p is the result of Newtonian mechanics for the Planck aether, but we still have to show that the same Schrödinger equation is valid for a mass $m \neq m_p$. To prove this generalization, we remark that the kinetic energy entering the Schrödinger equation (3) implies a zero point energy $(1/2)\hbar\omega_p$, where $\omega_p = c/r_p$, is the Planck frequency. For the Schrödinger equation to be valid for a mass $m \neq m_p$, this mass m would have to be subject to a zero point energy $(1/2)\hbar\omega$, where $\omega = c/\lambda_c$, with $\lambda_c = \hbar/mc$, instead of $\lambda_c = r_p = \hbar/m_p c$ as it is true for the zero point energy of the Planck masses. With $\omega = c/\lambda_c$ the zero point energy of a mass m is given by $E = (1/2)mc^2$, and the frequency associated with this zero point energy is $\omega = mc^2/\hbar$, which scales as the particle mass. In the Planck aether model wavelike disturbances are propagated with the velocity of light, establishing Lorentz invariance as a derived dynamic symmetry. It thus follows that the zero point energy spectrum must be proportional to ω^3, the only one invariant under a Lorentz transformation. With $4\pi\omega^2 d\omega$ states in frequency space located between ω and $\omega + d\omega$, the zero point energy of each mode must then just be proportional to ω, and hence to m, to obtain the required ω^3 dependence. We are therefore led to the strange conclusion that it is Lorentz invariance which assures that the nonrelativistic Schrödinger equation, derived from the Newtonian mechanics of the Planck masses, remains valid for masses $m \neq m_p$. However, because the zero point energy spectrum has a cut off at $\omega = \omega_p$, the Schrödinger equation would be valid only for masses $m \lesssim m_p$, to be replaced by Newtonian mechanics for masses $m \gg m_p$.

We would like to remark that a model to explain quantum mechanics by classical Newtonian mechanics was previously proposed by Weizel [4]. In Weizel's model the "Zitterbewegung" was caused through collisions with hypothetical particles, called "zerons". Weizel's model was criticized by Heisenberg [5] on the grounds that it violated the second law of thermodynamics for the zerons, which through the collisions would lead to an increase in entropy, contradicting the reversibility of the Schödinger equation. A somewhat similar model was later proposed by Nelson [6], which should be subject to the same criticism. Because collisions between positive and negative masses only lead to a dispersion of the particle trajectories, with the particle velocities remaining the same in both magnitude and direction, Heisenberg's objection does not apply if the dispersion is caused through the collision with negative mass particles.

Finally, we would like to remark that the presented derivation shows that quantum mechanics has its cause in the existence of negative masses. It also explains the time-energy uncertainty relation, because the existence of negative masses permits positive masses to borrow for a short time energy from the negative masses.

Appendix

To obtain an expression for the collision integral taking into account higher order terms otherwise suppressed by the Planck length, we have (by assuming that $f_\pm \simeq f_\mp$)

$$\begin{aligned}\log f'_\pm f'_\mp &= \log f_\pm\left(\boldsymbol{r} \pm \frac{\boldsymbol{r}_p}{2}\right) + \log f_\mp\left(\boldsymbol{r} \mp \frac{\boldsymbol{r}_p}{2}\right)\\ &\simeq \log f_\pm\left(\boldsymbol{r} \pm \frac{\boldsymbol{r}_p}{2}\right) + \log f_\pm\left(\boldsymbol{r} \mp \frac{\boldsymbol{r}_p}{2}\right)\\ &= \exp\left(\pm\frac{\boldsymbol{r}_p}{2}\cdot\frac{\partial}{\partial \boldsymbol{r}}\right)[\log f_\pm]\\ &+ \exp\left(\mp\frac{\boldsymbol{r}_p}{2}\cdot\frac{\partial}{\partial \boldsymbol{r}}\right)[\log f_\pm]\\ &= 2\cosh\left(\pm\frac{\boldsymbol{r}_p}{2}\cdot\frac{\partial}{\partial \boldsymbol{r}}\right)[\log f_\pm], \end{aligned} \tag{A.1}$$

where $\frac{\boldsymbol{r}_p}{2}\cdot\frac{\partial}{\partial \boldsymbol{r}}$ is an operator for which $\left(\frac{\boldsymbol{r}_p}{2}\cdot\frac{\partial}{\partial \boldsymbol{r}}\right)^n = \left(\frac{\boldsymbol{r}_p}{2}\right)^n \cdot \frac{\partial^n}{\partial \boldsymbol{r}^n}$. We thus have

$$f'_\pm f'_\mp - f_\pm f_\mp \simeq \exp\left\{2\cosh\left(\pm\frac{\boldsymbol{r}_p}{2}\cdot\frac{\partial}{\partial \boldsymbol{r}}\right)[\log f_\pm]\right\} - f_\pm^2. \tag{A.2}$$

To obtain from (A.2) the approximate expression (16), we expand the hyperbolic function up to the second order and the exponential function up to first order:

$$\begin{aligned}\exp\{\} - f_\pm^2 &= \exp\left\{\log f_\pm^2 + \left(\frac{\boldsymbol{r}_p}{2}\right)^2 \frac{\partial^2 \log f_\pm}{\partial \boldsymbol{r}^2}\right\} - f_\pm^2\\ &\simeq \left(\frac{\boldsymbol{r}_p}{2}\right)^2 f_\pm^2 \frac{\partial^2 \log f_\pm}{\partial \boldsymbol{r}^2}, \end{aligned} \tag{A.3}$$

which is the same as (16).

Inserting (A.2) into (11), putting $\alpha = 1$, applying the averaging operator $(\boldsymbol{r}_p/2)c\,\partial^2/\partial \boldsymbol{v}_\pm \partial \boldsymbol{r}$, and finally integrating over $d\boldsymbol{v}_\mp$, whereby $\int f_\mp \, d\boldsymbol{v}_\mp \simeq 1/2 r_p^3$, one has

$$\frac{\partial f_\pm}{\partial t} + \boldsymbol{v}_\pm \cdot \frac{\partial f_\pm}{\partial \boldsymbol{r}} \mp \frac{1}{m_p}\frac{\partial U}{\partial \boldsymbol{r}} \cdot \frac{\partial f_\pm}{\partial \boldsymbol{v}_\pm} = -c^2 \frac{\partial^2}{\partial \boldsymbol{v}_\pm \, \partial \boldsymbol{r}} \cdot \left\{ f_\pm \left[\frac{\exp\left\{ 2\cosh\left(\pm \frac{\boldsymbol{r}_p}{2} \cdot \frac{\partial}{\partial \boldsymbol{r}} \right) [\log f_\pm] \right\}}{f_\pm^2} - 1 \right] \right\} \quad \text{(A.4)}$$

Integrating (A.4) over $d\boldsymbol{v}_\pm$ one obtains the continuity equation (18). Multiplying (A.4) by $\boldsymbol{v}_\pm$, integrating over $d\boldsymbol{v}_\pm$, and setting $\partial^2 \log f_\pm / \partial \boldsymbol{r}^2 \simeq \partial^2 \log n_\pm / \partial \boldsymbol{r}^2$, one obtains

$$\frac{\partial (n_\pm \boldsymbol{V}_\pm)}{\partial t} + \frac{\partial (n_\pm \boldsymbol{V}_\pm \cdot \boldsymbol{V}_\pm)}{\partial \boldsymbol{r}} = \mp \frac{n_\pm}{m_p} \frac{\partial U}{\partial \boldsymbol{r}} + c^2 \frac{\partial}{\partial \boldsymbol{r}} \cdot \left\{ n_\pm \left[\frac{\exp\left\{ 2\cosh\left(\pm \frac{\boldsymbol{r}_p}{2} \cdot \frac{\partial}{\partial \boldsymbol{r}} \right) [\log n_\pm] \right\}}{n_\pm^2} - 1 \right] \right\}, \quad \text{(A.5)}$$

which by (18) can be simplified:

$$\frac{\partial \boldsymbol{V}_\pm}{\partial t} + \boldsymbol{V}_\pm \cdot \frac{\partial \boldsymbol{V}_\pm}{\partial \boldsymbol{r}} = \mp \frac{1}{m_p} \frac{\partial U}{\partial \boldsymbol{r}} + \frac{c^2}{n_\pm} \frac{\partial}{\partial \boldsymbol{r}} \cdot \left\{ n_\pm \left[\frac{\exp\left\{ 2\cosh\left(\pm \frac{\boldsymbol{r}_p}{2} \cdot \frac{\partial}{\partial \boldsymbol{r}} \right) [\log n_\pm] \right\}}{n_\pm^2} - 1 \right] \right\}. \quad \text{(A.6)}$$

By applying the inverted Madelung transformation on (A.6) one can obtain higher order correction terms for the Schrödinger equation, including nonlinear terms suppressed by the Planck length.

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